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Constructive method for solvability of Fredholm equation of the first kind

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Abstract. The solvability and construction of the general solution of the first kind Fredholm integral equation are among the insufficiently explored problems in mathematics. There are various approaches to solve this problem. We note the following methods for solving of ill-posed problem: regularization method, the method of successive approximations, the method of undetermined coefficients. The purpose of this work is to create a new method for solvability and construction of solution of the integral equation of the first kind. As it follows from the foregoing, the study of the solvability and construction of a solution of the Fredholm integral equation of the first kind is topical. A new method for studying of solvability and construction of a solution for Fredholm integral equation of the first kind is proposed. Solvability conditions and the construction method of an approximate solution of the integral Fredholm equation of the first kind are obtained.

Keywords: integral equation, solvability, construction of a solution, extreme problem, functional gradient, minimizing sequences.

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1 Introduction

Solving of the controllability problems of dynamical system [1, 6, 8], the mathematical theory of optimal processes [2, 10, 11], the boundary value problems of differential equations with phase and integral constraints [7, 12, 14] are reduced to solvability and construction of the general solution of the first kind integral equation

$$Ku = \int_{t_0}^{t_1} K(t,\tau)u(\tau)d\tau = f(t), \tag{1.1}$$

where $K(t,\tau)$ is a measurable function on the set $S_0 = \{(t,\tau) \in \mathbb{R}^2 \ / \ t_0 \le t \le t_1, \ \ t_0 \le \tau \le t_1\}$ and there exists an integral

$$P^2 = \int_{t_0}^{t_1}\int_{t_0}^{t_1} |K(t,\tau)|^2 dt d\tau < \infty,$$

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function $f(t) \in L_2(I, \mathbb{R}^1)$. It is necessary to find a solution $u(\tau) \in L_2(I, \mathbb{R}^1)$, where $I = [t_0, t_1]$.

The solvability and construction of the general solution of Fredholm integral equation of the first kind is related to insufficiently explored problems in mathematics.

As it follows from [22], the norm $\|K\| \leq P$, the operator K with kernel from $L_2(S_0)$ is a completely continuous operator which transfers every weakly convergent sequence into strongly convergent. The inverse operator is not limited [17], the equation $Ku = f$ can not be solvable for all $f \in L_2$. This leads to the fact that a small error in f leads to an arbitrarily large error in solution of the equation (1.1).

The famous theoretical results on the solvability of equation (1.1) refer to the case when $K(t, \tau) = K(\tau, t)$ i.e. equation (1.1) with a symmetric kernel. One of the main results of the solvability of equation (1.1) is a Picard theorem [18]. However, for application of this theorem necessary to prove the completeness of the eigenfunctions of symmetric kernel.

Thus, solvability and construction of solution of the integral equation (1.1) is a few studied complex ill-posed problem. There are various approaches to solve this problem. Note the following methods for solving of ill-posed problem.

- The regularization method [19] based on reducing the original problem to a correct problem. For regularization it is necessary to perform a priori requirements to the original data of the problem. In the works [16,20] the methods for solving of the correct problem after regularization are proposed. Unfortunately, the additional requirements imposed to the original data of the problem are not always held and the methods of solving the correct problem are time-consuming;

- The method of successive approximations [15] for solving of the equation (1.1). The method is applicable when $K(t, \tau)$ is a symmetric positive kernel in L_2 and it is required definition of the least characteristic number;

- The method of undetermined coefficients [21]. It is proposed to seek solutions to the equation (1.1) as a series. However, in general, the determination of the coefficients is extremely difficult.

The solvability and construction of solution of the equation (1.1) is topical as it follows from the study above.

The purpose of this work is to create a new method for construction and solvability of a solution of the integral equation of the first kind.

2 Solvability of Fredholm integral equation of the first kind

Problem statement. We consider the integral equation in the form

$$Ku = \int_a^b K(t, \tau) u(\tau) d\tau = f(t), \qquad t \in [t_0, t_1] = I, \tag{2.1}$$

where $K(t, \tau) = \|K_{ij}(t, \tau)\|$, $i = \overline{1, n}$, $j = \overline{1, m}$ is known matrix of $n \times m$ order, elements of the matrix $K(t, \tau)$ are $K_{ij}(t, \tau)$ which are measurable functions and belong to the class L_2 on the set $S_1 = \{(t, \tau) \in \mathbb{R}^2 \ / \ t_0 \leq t \leq t_1, \ a \leq \tau \leq b\}$,

$$\int_a^b \int_{t_0}^{t_1} |K_{ij}(t, \tau)|^2 dt d\tau < \infty,$$

function $f(t) \in L_2(I, \mathbb{R}^n)$ is prescribed, $u(\tau) \in L_2(I_1, \mathbb{R}^m)$ is the original function with $I_1 = [a, b]$, the values t_0, t_1, a, b are fixed, $K : L_2(I_1, \mathbb{R}^m) \to L_2(I, \mathbb{R}^n)$.

The following problems are set.

Problem 2.1. *Find necessary and sufficient conditions for existence of a solution of the integral equation (2.1) for a given $f(t) \in L_2(I, \mathbb{R}^n)$.*

Problem 2.2. *Find a solution of the integral equation (2.1) for a given $f(t) \in L_2(I, \mathbb{R}^n)$.*

Problem 2.3. *Find necessary and sufficient conditions for existence of a solution of the integral equation (2.1) for a given $f(t) \in L_2(I, \mathbb{R}^n)$, when the original function $u(\tau) \in U(\tau) \subset L_2(I_1, \mathbb{R}^m)$.*

Problem 2.4. *Find a solution of the integral equation (2.1) for a given $f(t) \in L_2(I_1, \mathbb{R}^m)$, when $u(\tau) \in U(\tau) \subset L_2(I_1, \mathbb{R}^m)$.*

Problem 2.5. *Find an approximate solution of the integral equation (2.1).*

As it follows from the problem statement the solvability and construction of the solution of matrix Fredholm integral equation of the first kind are considered. As well as construction of an approximate solution of Fredholm integral equation of the first kind. The results are correct for the matrix Fredholm integral equation of the first kind, as with non symmetrical, as symmetrical kernel.

This work is a continuation of the research presented in [1,2,6–8,10–12,14], [3–5,9,13].

We consider the solutions of the Problems 2.1, 2.2 for integral equation (2.1). The solutions of Problems 2.1 and 2.2 can be reduced to the study of extreme problem: minimize the functional

$$J(u) = \int_{t_0}^{t_1} \left| f(t) - \int_a^b K(t,\tau) u(\tau) d\tau \right|^2 dt \to \inf, \tag{2.2}$$

at condition

$$u(\tau) \in L_1(I_1, \mathbb{R}^m), \tag{2.3}$$

where $f(t) \in L_2(I, \mathbb{R}^n)$ is a prescribed function, $|\cdot|$ is Euclidean norm.

Theorem 2.6. *Let the kernel of the operator $K(t, \tau)$ be measurable and belongs to the class L_2 in the rectangle $S_1 = \{(t, \tau) \ / \ t \in I = [t_0, t_1], \ \ \tau \in I_1 = [a, b]\}$.*

Then:

(i) the functional (2.2) at condition (2.3) is continuously Fréchet differentiable, the gradient of the functional $J'(u) \in L_2(I_1, \mathbb{R}^m)$ at any point $u(\cdot) \in L_2(I_1, \mathbb{R}^m)$ is defined by formula

$$J'(u) = -2 \int_{t_0}^{t_1} K(t,\tau) f(t) dt + 2 \int_{t_0}^{t_1} \int_a^b K^*(t,\tau) K(t,\sigma) u(\sigma) d\sigma dt \in L_2(I_1, \mathbb{R}^m); \tag{2.4}$$

(ii) the gradient of the functional $J'(u) \in L_2(I_1, \mathbb{R}^m)$ satisfies to the Lipschitz condition

$$\|J'(u+h) - J'(u)\| \le l\|h\|, \qquad \forall u, \ u+h \in L_2(I_1, \mathbb{R}^m); \tag{2.5}$$

(iii) the functional (2.2) at condition (2.3) is convex, i.e.

$$J(\alpha u + (1-\alpha) v) \le \alpha J(u) + (1-\alpha) J(v), \qquad \forall u, v \in L_2(I_1, \mathbb{R}^m), \ \forall \alpha, \ \alpha \in [0,1]; \tag{2.6}$$

(iv) the second Fréchet derivative equals

$$J''(u) = 2\int_{t_0}^{t_1} K^*(t,\sigma)K(t,\tau)dt. \tag{2.7}$$

(v) if the inequality is satisfied

$$\begin{aligned}\int_a^b \int_a^b \xi^*(\sigma)\left[\int_{t_0}^{t_1} K^*(t,\sigma)K(t,\tau)dt\right]\xi(\tau)d\tau d\sigma = \int_{t_0}^{t_1}\left[\int_{t_0}^{t_1} K(t,\tau)\xi(\tau)d\tau\right]^2 dt \\ \geq \mu\int_a^b |\xi(\tau)|^2 d\tau, \qquad \mu > 0,\ \forall\xi,\ \xi \in L_2(I_1,\mathbb{R}^m),\end{aligned} \tag{2.8}$$

then the functional (2.2) at condition (2.3) is strongly convex.

Proof. As it follows from (2.2) the functional

$$J(u) = \int_{t_0}^{t_1}\left[f^*(t)f(t) - 2f^*(t)\int_a^b K(t,\tau)u(\tau)d\tau + \int_a^b\int_a^b u^*(\tau)K^*(t,\tau)K(t,\sigma)u(\sigma)d\sigma\right]dt.$$

Then the increment of the functional

$$\begin{aligned}\Delta J &= J(u+h) - J(u)\\ &= \int_a^b \left\langle -2\int_{t_0}^{t_1} K^*(t,\sigma)f(t)dt,\ h(\sigma)\right\rangle d\sigma\\ &\quad + \int_a^b \left\langle 2\int_{t_0}^{t_1}\int_a^b K^*(t,\sigma)K(t,\tau)u(\tau)d\tau dt,\ h(\sigma)\right\rangle d\sigma\\ &\quad + \int_{t_0}^{t_1}\int_a^b\int_a^b h^*(\tau)K^*(t,\tau)K(t,\sigma)h(\sigma)d\sigma d\tau dt\\ &= \langle J'(u), h\rangle_{L_2} + o(h),\end{aligned} \tag{2.9}$$

where

$$|o(h)| = \left|\int_{t_0}^{t_1}\left[\int_a^b\int_a^b h^*(\tau)K^*(t,\tau)K(t,\sigma)h*(\sigma)d\sigma d\tau\right]\right| dt \leq c_1\|h\|^2_{L_2}.$$

From (2.9) follows, that $J'(u)$ is defined by formula (2.4). Since

$$J'(u+h) - J'(u) = 2\int_{t_0}^{t_1}\int_a^b K^*(t,\tau)K(t,\sigma)h(t,\sigma)d\sigma dt,$$

that

$$\begin{aligned}|J'(u+h) - J'(u)| &\leq 2\int_{t_0}^{t_1}\int_a^b \|K^*(t,\tau)\|\ \|K(t,\sigma)\|\ |h(t,\sigma)|d\sigma dt\\ &\leq c_2(\tau)\|h\|_{L_2}, \qquad c_2(\tau) > 0,\ \ \tau \in I_1.\end{aligned}$$

Then

$$\|J'(u+h) - J'(u)\|_{L_2} = \left(\int_{t_0}^{t_1} |J'(u+h) - J'(u)|^2 d\tau\right)^{1/2} \leq l\|h\|_{L_2},$$

for any u, $u+h \in L_2(I_1,\mathbb{R}^m)$. This implies the inequality (2.5).

We show, that the functional (2.2) at condition (2.3) is convex. In fact, for every $u,\ w \in L_2(I_1, \mathbb{R}^m)$ the following inequality is valid:

$$\begin{aligned}
&\langle J'(u) - J'(w), u - w\rangle_{L_2} \\
&= \left\langle 2\int_{t_0}^{t_1}\int_a^b K^*(t,\tau)K(t,\sigma)[u(\sigma) - w(\sigma)]d\sigma dt,\ u - w\right\rangle_{L_2} \\
&= 2\int_{t_0}^{t_1}\left\{\int_a^b\int_a^b [u(\sigma) - w(\sigma)]^* K^*(t,\tau)K(t,\sigma)[u(\sigma) - w(\sigma)]d\sigma dt\right\} d\tau \\
&= 2\int_{t_0}^{t_1}\left[\int_a^b K(t,\sigma)[u(\sigma) - w(\sigma)]d\sigma\right]^2 dt \geq 0.
\end{aligned}$$

This means that the functional (2.2) is convex, i.e. the inequality (2.6) holds. As it follows from (2.4),

$$\begin{aligned}
J'(u+h) - J'(u) = \langle J''(u), h\rangle &= \left\langle 2\int_{t_0}^{t_1} K^*(t,\sigma)K(t,\tau)dt,\ h\right\rangle_{L_2} \\
&= 2\int_{t_0}^{t_1} K^*(t,\tau)K(t,\sigma)h(\sigma)d\sigma dt.
\end{aligned}$$

Consequently, $J''(u)$ is defined by formula (2.7). From (2.7), (2.8) follows that

$$\langle J''(u)\xi, \xi\rangle_{L_2} \geq \mu\|\xi\|^2, \qquad \forall u,\ u \in L_2(I_1, \mathbb{R}^m),\ \ \forall \xi,\ \xi \in L_2(I_1, \mathbb{R}^m).$$

This means that the functional $J(u)$ is strongly convex in $L_2(I_1, \mathbb{R}^m)$. The theorem is proved. □

Theorem 2.7. *Let for extreme problem (2.2), (2.3) the sequence $\{u_n(\tau)\} \in L_2(I_1, \mathbb{R}^n)$ be constructed by algorithm [5]*

$$u_{n+1}(\tau) = u_n(\tau) - \alpha_n J'(u_n), \qquad g_n(\alpha_n) = \min g_n(\alpha),\ \ \alpha \geq 0,$$

$$g_n(\alpha) = J(u_n - \alpha J'(u_n)), \qquad n = 0, 1, 2, \ldots$$

Then the numerical sequence $\{J(u_n)\}$ decreases monotonically, the limit $\lim_{n\to\infty} J'(u_n) = 0$. If, in addition, the set $M(u_0) = \{u(\cdot) \in L_2(I_1, \mathbb{R}^n) / J(u) \leq J(u_0)\}$ is bounded, then:

(i) the sequence $\{u_n(\tau)\} \subset M(u_0)$ is minimized, i.e.

$$\lim_{n\to\infty} J(u_n) = J_* = \inf J(u), \qquad u \in L_2(I, \mathbb{R}^m);$$

(ii) the sequence $\{u_n\}$ weakly converges to the set U_*, where

$$U_* = \left\{u_*(\tau) \in L_2(I_1, \mathbb{R}^m) / J(u_*) = \min_{u\in M(u_0)} J(u) = J_* = \inf_{u\in L_2(I_1,\mathbb{R}^m)} J(u)\right\},$$

$u_n \overset{weak}{\to} u_*$ as $n \to \infty$;

(iii) the following rate of convergence is valid

$$0 \leq J(u_n) - J(u_*) \leq \frac{m_0}{n}, \qquad m_0 = \text{const.} > 0,\ \ n = 1, 2, \ldots \tag{2.10}$$

(iv) *if the inequality* (2.8) *is satisfied, then the sequence* $\{u_n\} \subset L_2(I_1, \mathbb{R}^n)$ *strongly converges to the point* $u_* \in U_*$. *The following estimates are valid*

$$0 \le J(u_n) - J(u_*) \le [J(u_0) - J_*]q^n, \qquad q = 1 - \frac{\mu}{l}, \;\; 0 \le q \le 1, \;\; \mu > 0,$$
$$\|u_n - u_*\| \le \left(\frac{2}{\mu}\right) [J(u_0) - J(u_*)]\, q^n, \qquad n = 0, 1, 2, \dots, \tag{2.11}$$

where $J_* = J(u_*)$;

(v) *in order that the Fredholm integral equation of the first kind* (2.1) *have a solution it is necessary and sufficient that* $J(u_*) = 0$, $u_* \in U_*$. *In this case the function* $u_*(\tau) \in L_2(I_1, \mathbb{R}^m)$ *is a solution of integral equation* (2.1).

(vi) *if the value* $J(u_*) > 0$, *then the integral equation* (2.1) *has no solution for a given* $f(t) \in L_2(I, \mathbb{R}^n)$.

Proof. Since $g_n(\alpha_n) \le g_n(\alpha)$, that $J(u_n) - J(u_{n+1}) \ge J(u_n) - J(u_n - \alpha J'(u_n))$, $\alpha \ge 0$, $n = 0, 1, 2, \dots$ On the other hand, from the inclusion $J(u) \in C^{1,1}(L_2(I_1, \mathbb{R}^m))$ follows, that

$$J(u_n) - J(u_n - \alpha J'(u_n)) \ge \alpha \left(1 - \frac{\alpha l}{2}\right) \|J'(u_n)\|^2, \qquad \alpha \ge 0, \;\; n - 0, 1, 2, \dots$$

Then

$$J(u_n) - J(u_{n+1}) \ge \frac{1}{2l} \|J'(u_n)\|^2 > 0.$$

It follows that the numerical sequence $\{J(u_n)\}$ decreases monotonically and $\lim_{n\to\infty} J'(u_n) = 0$. The first statement of the theorem is proved.

Since the functional $J(u)$ is convex at $u \in L_2$, that the set $M(u_0)$ is convex. Then

$$0 \le J(u_n) - J(u_*) \le \langle J'(u_n), u_n - u_* \rangle_{L_2} \le \|J'(u_n)\| \, \|u_n - u_*\| \le D\|J'(u_n)\|,$$

$$u_n \in M(u_0), \;\; u_* \in M(u_0),$$

where D is a diameter of the set $M(u_0)$. Since $M(u_0)$ is a bounded convex closed set in L_2, it is weakly bicompact. The convex continuously differentiable functional $J(u)$ is weakly lower semicontinuous. Then the set $U_* \ne \varnothing$, $\varnothing$ is the empty set, $U_* \subset M(u_0)$, $\{u_n\} \subset M(u_0)$, $u_* \in M(u_0)$. We note, that

$$0 \le \lim_{n\to\infty} J(u_n) - J(u_*) \le D \lim_{n\to\infty} \|J'(u_n)\| = 0, \qquad \lim_{n\to\infty} J(u_n) = J(u_*) = J_*.$$

Consequently, on the set $M(u_0)$ the lower bound of the functional $J(u)$ in the point $u_* \in U_*$ is reached, the sequence $\{u_n\} \subset M(u_0)$ is minimized. Thus, the second statement of the theorem is proved.

The third statement of the theorem follows from the inclusion $\{u_n\} \subset M(u_0)$, $M(u_0)$ is a weakly bicompact set, $J(u_*) = \min J(u) = J_* = \inf J(u)$, $u \in M(u_0)$. Therefore, $u_n \overset{\text{weak}}{\to} u_*$ at $n \to \infty$.

From the inequalities

$$J(u_n) - J(u_{n+1}) \le \frac{1}{2l} \|J'(u_n)\|^2, \qquad 0 \le J(u_n) - J(u_*) \le D\|J'(u_n)\|,$$

$$u_n \overset{\text{weak}}{\to} u_* \quad \text{as } n \to \infty.$$

follows the estimation (2.10), where $m_0 = 2D^2 l$. The fourth statement of the theorem is proved.

If the inequality (2.8) holds, then the functional (2.2) at condition (2.3) is strongly convex. Then

$$J(u_n) - J(u_*) \leq \langle J'(u_n), u_n - u_* \rangle - \frac{\mu}{2}\|u_n - u_*\|^2 \leq 2\mu\|J'(u_n)\|^2, \qquad n = 0, 1, 2, \ldots,$$

$$J(u_n) - J(u_{n+1}) \geq \frac{1}{2l}\|J'(u_n)\|^2, \qquad n = 0, 1, 2, \ldots$$

It follows that $a_n - a_{n+1} \geq \frac{\mu}{l} a_n$, where $a_n = J(u_n) - J(u_*)$. Therefore, $0 \leq a_{n+1} \leq a_n(1 - \frac{\mu}{l}) = q a_n$. Then $a_n \leq q a_{n-1} \leq q^2 a_{n-2} \leq \cdots \leq q^n a_0$, where $a_0 = J(u_0) - J(u_*)$. It follows the estimations (2.11). The fifth statement of the theorem is proved.

As it follows from (2.2), the value $J(u) \geq 0$, $\forall u$, $u \in L_2(I_1, \mathbb{R}^m)$. Sequence $\{u_n\} \subset L_2(I_1, \mathbb{R}^m)$ is minimizing for any starting point $u_0 = u_0(\tau) \in L_2(I_1, \mathbb{R}^m)$, i.e.

$$J(u_*) = \min_{u \in L_2(I_1, \mathbb{R}^m)} J(u) = J_* = \inf_{u \in L_2(I_1, \mathbb{R}^m)} J(u).$$

If $J(u_*) = 0$, then $f(t) = \int_a^b K(t,\tau) u_*(\tau) d\tau$. Thus, the integral equation (2.1) has a solution if and only if the value $J(u_*) = 0$, where $u_* = u_*(\tau) \in L_2(I_p, \mathbb{R}^n)$ is solution of integral equation (2.1). If the value $J(u_*) > 0$, then $f(t) \neq \int_a^b K(t,\tau) u_*(\tau) d\tau$, consequently, $u_* = u_*(\tau)$, $\tau \in I_1$ is not the solution of the integral equation (2.1). The theorem is proved. □

We consider the case when the original function $u(\tau) \in U(\tau) \subset L_2(I_1, \mathbb{R}^m)$, where, in particular, either

$$U(\tau) = \{u(\cdot) \in L_2(I_1, \mathbb{R}^m) / \alpha(\tau) \leq u(\tau) \leq \beta(\tau), \text{ a.e. } \tau \in I_1\},$$

or

$$U(\tau) = \{u(\cdot) \in L_2(I_1, \mathbb{R}^m) / \|u\|^2 \leq \mathbb{R}^2\}.$$

Solutions of the Problems 2.3 and 2.4 can be obtained by solving the optimization problem: minimize the functional

$$J_1(u, v) = \int_{t_0}^{t_1} |f(t) - \int_a^b K(t,\tau) u(\tau) d\tau|^2 dt + \|u - v\|_{L_2}^2 \to \inf \tag{2.12}$$

at condition

$$u(\cdot) \in L_2(I_1, \mathbb{R}^m), \; v(\tau) \in U(\tau) \subset L_2(I_1, \mathbb{R}^m), \; \tau \in I_1, \; f(t) \in L_2(I, \mathbb{R}^n). \tag{2.13}$$

Theorem 2.8. *Let the kernel of the operator $K(t,\tau)$ be measurable and belong to L_2 in the rectangle $S_1 = \{(t,\tau) \in \mathbb{R}^2 / t \in I, \; \tau \in I_1\}$. Then:*

(i) the functional (2.12) at condition (2.13) is continuously Fréchet differentiable, the gradient of the functional

$$J_1'(u, v) = (J_{1u}'(u, v), J_{1v}'(u, v)) \in L_2(I_1, \mathbb{R}^m) \times L_2(I_1, \mathbb{R}^m)$$

in any point $(u,v) \in L_2(I_1,\mathbb{R}^m) \times L_2(I_1,\mathbb{R}^m)$ *is defined by formula*

$$\begin{aligned} J'_{1u}(u,v) =& -2\int_{t_0}^{t_1} K(t,\tau)f(t)dt + 2\int_{t_0}^{t_1}\int_a^b K^*(t,\tau)K(t,\sigma)u(\sigma)d\sigma dt \\ &+ 2(u-v) \in L_2(I_1,\mathbb{R}^m), \\ J'_{1v}(u,v) =& -2(u-v) \in L_2(I_1,\mathbb{R}^m); \end{aligned} \tag{2.14}$$

(ii) the gradient of the functional $J'_1(u,v)$ *satisfies the Lipschitz condition*

$$\|J'_1(u+h,v+h_1) - J'_1(u,v)\| \le l_2(\|h\| + \|h_1\|),$$
$$\forall (u,v),\ \ (u+h,v+h_1) \in L_2(I_1,\mathbb{R}^m) \times L_2(I_1,\mathbb{R}^m);$$

(iii) the functional (2.12) *at condition* (2.13) *is convex.*

The proof of the theorem is similar to the proof of Theorem 2.6.

Theorem 2.9. *Let for optimization problem* (2.12), (2.13) *the sequences be constructed (see* (2.14)*)*

$$u_{n+1} = u_n - \alpha_n J'_{1u}(u_n,v_n), \qquad v_{n+1} = P_U[v_n - \alpha_n J'_1(u_n,v_n)], \qquad n = 0,1,2,\dots,$$

$$\varepsilon_0 \le \alpha \le \frac{2}{l_2 + 2\varepsilon_1}, \quad \varepsilon_0 > 0, \quad \varepsilon_1 > 0, \quad n = 0,1,2,\dots$$

Then the numerical sequence $\{J_1(u_n,v_n)\}$ *decreases monotonically, and* $\lim_{n\to\infty}\|u_n - u_{n+1}\| = 0$, $\lim_{n\to\infty}\|v_n - v_{n+1}\| = 0$ *hold.*

If, in addition, the set $M(u_0,v_0) = \{(u,v) \in L_2 \times U / J_1(u,v) \le J(u_0,v_0)\}$ *is bounded, then:*

(i) the sequence $\{u_n,v_n\} \subset M(u_0,v_0)$ *is minimized, i.e.*

$$\lim_{n\to\infty} J_1(u_n,v_n) = J_* = \inf J(u,v), \qquad (u,v) \in L_2 \times U;$$

(ii) $u_n \overset{weak}{\to} u_*$, $v_n \overset{weak}{\to} v_*$ *as* $n \to \infty$,

$$(u_*,v_*) \in U_* = \Big\{(u_*,v_*) \in L_2 \times U / J_1(u_*,v_*) = \min J_1(u,v) = J_* = \inf J_1(u,v),$$
$$(u,v) \in L_2 \times U\Big\};$$

(iii) in order that the integral equation (2.1) *at condition* $u(\tau) \in U$ *have a solution, it is necessary and sufficient that* $J_1(u_*,v_*) = 0$.

The proof of the theorem is similar to the proof of Theorem 2.7.

3 The approximate solution of Fredholm integral equation of the first kind

We consider the integral equation in the form

$$Ku = \int_a^b K(t,\tau)u(\tau)d\tau = f(t), \qquad t \in I = [t_0,t_1]. \tag{3.1}$$

Let in L_2 a complete system be given, such as $1, t, t^2, \dots$, and any complete orthonormal system $\{\varphi_k(t)\}_{k=1}^{\infty}$, $t \in I = [t_0, t_1]$. Since the condition of Fubini's theorem on changing the order of integration is held, that (see (3.1))

$$\begin{aligned}
&\int_{t0}^{t_1} \left(\int_a^b K_{ij}(t,\tau) u_j(\tau) d\tau \right) \varphi_k(t) dt \\
&\qquad = \int_a^b \left(\int_{t0}^{t_1} K_{ij}(t,\tau) \varphi_k(t) dt \right) u_j(\tau) d\tau \\
&\qquad = \int_a^b L_{ij}^{(k)}(\tau) u_j(\tau) d\tau, \qquad i = \overline{1,n}, \;\; j = \overline{1,m}, \; k = 1,2,\dots,
\end{aligned}$$

$$\int_{t0}^{t_1} f_i(t) \varphi_k(t) dt = a_{ik}, \qquad i = \overline{1,n}, \; k = 1,2,\dots,$$

where $K(t,\tau) = \|K_{ij}(t,\tau)\|$, $i = \overline{1,n}$, $j = \overline{1,m}$, $f(t) = (f_1(t), \dots, f_n(t))$, $t \in I$, $\tau \in I_1$, $L_{ij}^{(k)}(\tau)$ is denoted $\int_{t_0}^{t_1} K_{ij}(t,\tau)\varphi_k(t) dt$.

Then

$$\begin{aligned}
&\int_{t0}^{t_1} \left(\int_a^b K(t,\tau) u(\tau) d\tau \right) \varphi_k(t) dt \\
&\quad = \begin{pmatrix} \int_a^b \left(\int_{t0}^{t_1} K_{11}(t,\tau)\varphi_k(t)dt \right) u_1(\tau)d\tau + \cdots + \int_a^b \left(\int_{t0}^{t_1} K_{1m}(t,\tau)\varphi_k(t)dt \right) u_m(\tau)d\tau \\ \vdots \\ \int_a^b \left(\int_{t0}^{t_1} K_{n1}(t,\tau)\varphi_k(t)dt \right) u_1(\tau)d\tau + \cdots + \int_a^b \left(\int_{t0}^{t_1} K_{nm}(t,\tau)\varphi_k(t)dt \right) u_m(\tau)d\tau \end{pmatrix} \\
&\quad = \begin{pmatrix} \int_a^b L_{11}^{(k)}(\tau) u_1(\tau) d\tau + \cdots + \int_a^b L_{1m}^{(k)}(\tau) u_m(\tau) d\tau \\ \vdots \\ \int_a^b L_{n1}^{(k)}(\tau) u_1(\tau) d\tau + \cdots + \int_a^b L_{nm}^{(k)}(\tau) u_m(\tau) d\tau \end{pmatrix} \\
&\quad = \int_a^b L^{(k)}(\tau) u(\tau) d\tau, \qquad k = 1,2,\dots,
\end{aligned}$$

$$\overline{a}^{(k)} = \int_{t0}^{t_1} f(t)\varphi_k(t) dt = \begin{pmatrix} \int_{t0}^{t_1} f_1(t)\varphi_k(t)dt \\ \vdots \\ \int_{t0}^{t_1} f_n(t)\varphi_k(t)dt \end{pmatrix} = \begin{pmatrix} \overline{a}_1^{(k)} \\ \vdots \\ \overline{a}_n^{(k)} \end{pmatrix}, \qquad k = 1,2,\dots$$

Now, for each index k we get

$$\int_a^b L^{(k)}(\tau) u(\tau) d\tau = \overline{a}^{(k)}, \qquad k = 1,2,\dots, \tag{3.2}$$

where $L^{(k)}(\tau)$ is a matrix of $n \times m$ order, $\overline{a}^{(k)} \in R^n$,

$$L^{(k)}(\tau) = \begin{pmatrix} L_1^{(k)}(\tau) \\ \vdots \\ L_n^{(k)}(\tau) \end{pmatrix}, \qquad L_j^{(k)}(\tau) = \left(L_{j1}^{(k)}(\tau), \dots, L_{jm}^{(k)}(\tau) \right), \qquad k = 1,2,\dots$$

We denote

$$L(\tau) = \begin{pmatrix} L^{(1)}(\tau) \\ L^{(2)}(\tau) \\ \vdots \end{pmatrix}, \qquad \overline{a} = \begin{pmatrix} \overline{a}^{(1)} \\ \overline{a}^{(2)} \\ \vdots \end{pmatrix}.$$

Then the relations (3.2) are written in the form

$$\int_a^b L(\tau)u(\tau)d\tau = \overline{a}, \tag{3.3}$$

where $L(\tau)$ is a matrix of $Nn \times m$ order, $N = \infty$.

It should be noted that if for some $k = k_*$, $L_j^{(k_*)}(\tau) = 0$ and corresponding $\overline{a}_j^{(k_*)} = 0$, then the relations should be excepted from the system (3.3)

$$\int_a^b L_j^{(k_*)}(\tau)u(\tau)d\tau = \overline{a}_j^{(k_*)}.$$

Note that if $L_j^{(k_*)}(\tau) = 0$, but $a_j^{(k_*)} \neq 0$, then the integral equation (3.1) has no solution.

Theorem 3.1. *Let the matrix*

$$C_N = \int_a^b L_N(\tau)L_N^*(\tau)d\tau$$

of order $nN \times nN$ be positive definite. Then the general solution of the integral equation (3.1) *is determined by formula*

$$u_N(\tau) = L_N^*(\tau)C_N^{-1}\overline{a}_N + p_N(\tau) - L_N^*(\tau)C_N^{-1}\int_a^b L_N(\eta)p_N(\eta)d\eta, \qquad \tau \in I_1, \tag{3.4}$$

where $p_N(\tau) \in L_2(I_1, \mathbb{R}^m)$ is an arbitrary function.

The proof for finite N can be found in [3].

4 Conclusion

In this work a new method for studying of solvability and construction of a solution for Fredholm integral equation of the first kind is proposed. Necessary and sufficient conditions for existence of a solution for a given right-hand side are obtained in two cases: when the original function belongs to the space L_2; the original function belongs to a given set of L_2. Solvability conditions and the method of construction an approximate solution of the integral Fredholm equation of the first kind are obtained. According to the method with comparison to the well-known methods an approximation solution of the Fredholm integral equation of the first kind can be obtained. Several theorems about solvability of the equation are proved. Further continuation of the research works in this direction and development applications on the base of the method are planned.

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